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ADDENDUM

(q, h)-analogue of Newton's binomial formula

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Abstract. In this addendum, the (q, h)-analogue of Newton's binomial formula is obtained in the (q, h)-deformed quantum plane which reduces for $h = 0$ to the q-analogue. For $(q = 1, h = 0)$, this is just the usual one as it should be. Moreover, the h-analogue is recovered for $q = 1$. Some properties of the (q, h)-binomial coefficients are also given. This result will contribute to an introduction of the (q, h)-analogue of the well known functions, (q, h)-special functions and (q, h)-deformed analysis.

The q-analysis is an extension of the ordinary analysis, by the addition of an extra parameter q. When q tends towards one the usual analysis is recovered. Such q-analysis appeared in the literature a long time ago ‡. In particular, a q-analogue of Newton's binomial formula, well known functions like q-exponential, q-logarithm,... etc, the special functions arena's q-differentiation and q-integration have been introduced and studied intensively.

In [2], the h-analogue of Newton's binomial formula was introduced leading, therefore, to a new analysis, called h-analysis. In this addendum, I will go a step further by generalizing the work [2]. Indeed, an analogue of Newton's binomial formula is introduced here in the (q, h)-deformed quantum plane (i.e. (q, h) Newton's binomial formula which generalizes Schützenberger's formula [3] with an extra parameter h) leading, therefore, to a more generalized analysis, called (q, h)-analysis. With this generalization, the q-analysis, h-analysis and ordinary analysis are recovered respectively by taking $h = 0$, $q = 1$ and $(q = 1, h = 0)$.

Newton's binomial formula is defined as follows:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k} \tag{1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and it is understood here that the coordinate variables x and y commute, i.e. $xy = yx$. A q-analogue of (1), for the q-commuting coordinates x and y satisfying $xy = qyx$, first appeared in literature in Schützenberger [3], see also Cigler [4],

$$(x + y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k} \tag{2}$$

where the q-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

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‡ For historical details see [1]

with

$$[j]_q = \frac{1-q^j}{1-q}.$$

The h-analogue has been introduced and defined in [2] as follows:

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_h y^k x^{n-k} \tag{3}$$

provided that x and y satisfy to $[x, y] = hy^2$ and the h-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_h$ is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_h = \binom{n}{k} h^k (h^{-1})_k \tag{4}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the shifted factorial.

Now consider Manin's q-plane $x'y' = qy'x'$. By the following linear transformation (see [5] and references therein):

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \frac{h}{q-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Manin's q-plane changes to

$$xy = qyx + hy^2. \tag{5}$$

Even though the linear transformation is singular for $q = 1$, the resulting quantum plane is well-defined.

Proposition 1. *Let x and y be coordinate variables satisfying (5), then the following identities are true:*

$$\begin{aligned} x^k y &= \sum_{r=0}^{k} \frac{[k]_q!}{[k-r]_q!} q^{k-r} h^r y^{r+1} x^{k-r} \\ xy^k &= q^k y^k x + h\,[k]_q\; y^{k+1}. \end{aligned} \tag{6}$$

These identities are easily proved by successive use of (5).

Proposition 2 ((q, h)-binomial formula)). *Let x and y be coordinate variables satisfying (5), then we have*

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} y^k x^{n-k} \tag{7}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)}$ *are the (q, h)-binomial coefficients given as follows:*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} = \begin{bmatrix} n \\ k \end{bmatrix}_q h^k (h^{-1})_{[k]} \tag{8}$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_{(q,h)} = 1$ *and*

$$(a)_{[k]} = \prod_{j=0}^{k-1} (a + [j]_q) \tag{9}$$

since by definition $[0]_q = 0$.

Proof. This proposition will be proved by recurrence. Indeed for $n = 1, 2$, it is verified. Suppose now that the formula is true for $n - 1$, which means

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{(q,h)} y^k x^{n-1-k}$$

with $\begin{bmatrix} n-1 \\ 0 \end{bmatrix}_{(q,h)} = 1$. To show this for n, let us first consider the following expansion:

$$(x+y)^n = \sum_{k=0}^{n} C_{n,k} y^k x^{n-k}$$

where $C_{n,k}$ are coefficients depending on q and h.
Then, we have

$$(x+y)^n = (x+y)(x+y)^{n-1} = (x+y) \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{(q,h)} y^k x^{n-1-k}$$
$$= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{(q,h)} x y^k x^{n-1-k} + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{(q,h)} y^{k+1} x^{n-1-k}.$$

Using the result of the first proposition, we obtain

$$(x+y)^n = \begin{bmatrix} n-1 \\ 0 \end{bmatrix}_{(q,h)} + \sum_{k=1}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{(q,h)} q^k y^k x^{n-k}$$
$$+ \sum_{k=1}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{(q,h)} (1 + h[k]_q) y^{k+1} x^{n-1-k} + \begin{bmatrix} n-1 \\ 0 \end{bmatrix}_{(q,h)} y x^{n-1}$$

which yields, respectively,

$$C_{n,0} = \begin{bmatrix} n-1 \\ 0 \end{bmatrix}_{(q,h)} = 1$$
$$C_{n,1} = q \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_{(q,h)} + \begin{bmatrix} n-1 \\ 0 \end{bmatrix}_{(q,h)} = \begin{bmatrix} n \\ 1 \end{bmatrix}_{(q,h)}$$
$$C_{n,k} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{(q,h)} + (1 + h[k-1]_q) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{(q,h)} = \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)}$$
$$C_{n,n} = (1 + h[n-1]_q) \begin{bmatrix} n-1 \\ n-1 \end{bmatrix}_{(q,h)} = \begin{bmatrix} n \\ n \end{bmatrix}_{(q,h)}.$$

□

Moreover, the (q, h)-binomial coefficients obey the following properties $1 < k < n$:

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_{(q,h)} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} + (1 + h\,[k-1]_q) \begin{bmatrix} n \\ k-1 \end{bmatrix}_{(q,h)} \tag{10}$$

and

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{(q,h)} = (1 + h\,[k]_q) \frac{[n+1]_q}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)}. \tag{11}$$

In fact, these properties follow from the well known relations of the q-binomial coefficients:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \frac{[n+1]_q}{[k]_q} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

upon using $(a)_{[k]} = (a + [k-1]_q)(a)_{[k-1]}$.

Now, we make the following remarks. For $h = 0$ this is just the q-binomial formula as it should be. For $q = 1$, it reduces to the h-analogue Newton's binomial formula (3) and (4) and for $(q = 1, h = 0)$ the usual one is recovered.

To conclude, we have obtained a more general Newton's binomial formula in the (q, h)-deformed quantum plane which reduces to the known one at some limits. This will therefore lead to a more generalized analysis called (q, h)-analysis.

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References

[1] Exton H 1983 *q-Hypergeometric Functions and Applications* (Chichester: Ellis Horwood)
[2] Benaoum H B 1998 h-analogue of Newton's binomial formula *J. Phys. A: Math. Gen.* **31** L751
[3] Schützenberger M P 1953 Une interprétation de certaines solutions de l'équation fonctionnelle: $F(x + y) = F(x)F(y)$ *C. R. Acad. Sci. Paris* **236** 352
[4] Cigler J 1979 Operatormethoden für q-Identitäten *Monatsh. Math.* **88** 87
[5] Cho S, Madore J and Park K S 1998 Noncommutative geometry of the h-deformed quantum plane *J. Phys. A: Math. Gen.* **31** 2639